A VARIATIONAL INEQUALITY WITH MIXED BOUNDARY CONDITIONS

BY

M. K. V. MURTHY AND G. STAMPACCHIA†

ABSTRACT

In this paper we study a variational inequality for a second order uniformly elliptic operator on a bounded domain, the solution of which is required to lie above a given obstacle and to assume assigned values on a part of the boundary of the domain. We are mainly concerned with the regularity of the solution in relation to the regularity of the data.

This paper is concerned with a variational inequality for a linear second order uniformly elliptic operator A on a bounded domain Ω in $\mathbf{R}^n$, the solutions of which are required to assume assigned values only on a part of the boundary of the domain and to lie above a given obstacle.

The existence and uniqueness of the solution of the variational inequality under consideration is established in Section 1.

We show in Section 2 that, under very mild assumptions of smoothness on the domain Ω and on the coefficients of the operator A, the solution of the variational inequality, with an obstacle belonging to some $H^{1,p}(\Omega) \cap C^{0,\gamma}(\bar{\Omega})$, $p > n$ and very general data, is Hölder continuous up to the boundary with an exponent $0 < \lambda \leqq \gamma$ (λ depending on p).

It is shown in Section 3 that the solution of our variational inequality can be approximated by solutions of certain quasi-linear mixed boundary value problems associated with the given elliptic operator A. This procedure permits us to obtain further regularity results for the solution of this variational inequality.

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We note that the results of Peetre [16], Shamir [17] and others show that, however smooth the data, the domain and the coefficients of A may be, the solution of the mixed boundary value problem has an optimal regularity beyond which one cannot expect any smoothness unless some additional compatibility conditions are imposed. In connection with our variational inequality, higher regularity is impeded not only by this fact but also because we are not dealing with equations.

The concluding Section 4 is devoted to a formal interpretation of the boundary conditions imposed by the variational inequality. Also indicated are some extensions of our results to the corresponding questions associated with operators containing lower order terms and with inhomogeneous boundary values. Moreover, we also show that the problems in which the obstacle is defined only on the boundary as considered by Da Veiga [23] and Brézis [2], can be fitted into the framework of this paper by making use of an idea of Kinderlehrer [5].

1. Notations and statement of the problems

Let Ω be a bounded connected open set in the n-dimensional real Euclidean space $\mathbf{R}^n$, $\bar{\Omega}$ its closure and $\partial\Omega$ its boundary. We shall consider only real valued measurable functions and we shall use the following standard notation.

$C^k(\bar{\Omega})$, $0 \leqq k \leqq \infty$, denotes the space of all k-times continuously differentiable functions on $\bar{\Omega}$ and $\overset{k}{C_0}(\bar{\Omega})$ its subspace consisting of all functions with compact support in $\bar{\Omega}$, $C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, the space of all Hölder continuous functions on $\bar{\Omega}$ having Hölder exponent α. In a coordinate system $(x_1, \cdots, x_n)$, at a generic point x of $\bar{\Omega}$, the partial derivatives $\partial u / \partial x_j$ of a function u in $C^1(\bar{\Omega})$ will be denoted by u_{x_j} and its gradient $(u_{x_1}, \cdots, u_{x_n})$ by u_x. In the sequel, we shall use the summation convention that the sum is to be understood whenever an index appears repeated. We define also the scalar function

$$|u_x| = \Big(\sum_j u_{x_j}^2 \Big)^{\frac{1}{2}}.$$

We denote, for any $p \geqq 1$, the norm of a function u in $L^p(E)$ by $\| u \|_{p,E}$ or simply by $\| u \|_p$ when the domain of integration E is clear from the context.

We consider on the space $C^1(\bar{\Omega})$ the norm

$$\| u \|_{1,p} = \Big(\| u \|_{p,\Omega}^2 + \sum_j \| u_x \|_{p,\Omega}^2 \Big)^{\frac{1}{2}}. \tag{1.1}$$

$H^{1,p}(\Omega)$ denotes the Sobolev space of all distributions on Ω obtained by the completion of $C^1(\bar{\Omega})$ with respect to the norm (1.1). This is a Banach space (reflexive for $1 < p < \infty$). For $p = 2$ it is a Hilbert space, denoted simply by $H^1(\Omega)$, provided with the natural scalar product

$$(1.2) \qquad (u, v)_{1,2} = (u, v)_{L^2(\Omega)} + (u_{x_j}, v_{x_j})_{L^2(\Omega)}.$$

Here $\| u_x \|_{p,\Omega}$ is the norm in $L^p(\Omega)$ of the scalar function $|u_x|$. For $u \in H^{1,p}(\Omega)$, u_{x_j} and u_x still denote the derivative and the gradient in the sense of distributions.

If Ω satisfies a cone condition, then the following well-known Sobolev lemma holds: Every $u \in H^{1,p}(\Omega)$ for $1 < p < n$ belongs to $L^{p^*}(\Omega)$ where $p^{*-1} = p^{-1} - n^{-1}$ and there exists a constant $C > 0$ such that

$$(1.3) \qquad \| u \|_{p^*} \leqq C \| u \|_{1,p} \text{ for all } u \in H^{1,p}(\Omega).$$

We shall constantly make use of the following notions.

If E is any subset of $\bar{\Omega}$, we say that a distribution $u \in H^{1,p}(\Omega)$ vanishes on E if there exists a sequence $u_\nu \in C^1(\bar{\Omega})$ such that $u_\nu = 0$ on E and $u_\nu \to u$ in $H^{1,p}(\Omega)$. We say that $u \geqq 0$ on E if there exists a sequence $u_\nu \in C^1(\bar{\Omega})$ such that $u_\nu \geqq 0$ on E and $u_\nu \to u$ in $H^{1,p}(\Omega)$.

In the following, we shall denote by $|E|$ the n-dimensional and by $[E]$ the $(n-1)$-dimensional Hausdorff measure of the set E.

We shall denote by $H^{2,p}(\Omega)$ the space of all functions whose first derivatives belong to $H^{1,p}(\Omega)$.

Suppose now that $\partial_1\Omega$ and $\partial_2\Omega$ are two disjoint open subsets of the boundary $\partial\Omega$ such that $\partial\Omega = \overline{\partial_1\Omega} \cup \overline{\partial_2\Omega}$.

We shall denote by V the subspace of $H^1(\Omega)$ consisting of all distributions $u \in H^1(\Omega)$ such that $u = 0$ on $\partial_1\Omega$ (in the sense defined above). The space V provided with the norm induced from that of $H^1(\Omega)$, being a closed subspace, becomes a Hilbert space. We note first of all that $C^1(\bar{\Omega}) \cap V$ is dense in V.

If $\partial_1\Omega$ is locally Lipschitz, then we can define the trace of a distribution u in $H^{1,p}(\Omega)$ on $\partial_1\Omega$ and, by a well-known result concerning Sobolev spaces, this trace belongs to $L^s(\partial_1\Omega)$ where $s = p(n-1)/(n-p)$. It is then clear that V is precisely the space of all u in $H^1(\Omega)$ whose trace on $\partial_1\Omega$ is zero.

In the following, we shall assume that Ω and $\partial_1\Omega$ are such that the following Poincaré type inequality holds for all $u \in V$: There exists a constant $C = C(\Omega, \partial_1\Omega) > 0$ such that

$$(1.4) \qquad \| u \|_{2,\Omega} \leqq C \| u_x \|_{2,\Omega}.$$

Under this assumption, we can take on V the equivalent norm defined by

$$\| u \|_V = \| u_x \|_{2,\Omega}. \tag{1.5}$$

A sufficient condition for inequality (1.4) to hold for all u in V can be described as follows (see, for instance, [20]): For a point x fixed in $\mathbf{R}^n$, any point $y \in \mathbf{R}^n$ can be represented in polar coordinates with origin at x as $y = x + r\xi$ where $r = |x - y|$ and $|\xi| = 1$.

For any $x \in \Omega$, let $\Sigma(x)$ denote the set of points ξ of the unit sphere such that if $y = x + r\xi \in \partial_1\Omega$ then the segment $\{x + \mathrm{tr}\xi; 0 < t < 1\}$ joining x to y lies entirely in Ω. Denote by $[\Sigma(x)]$ the $(n-1)$-dimensional measure of $\Sigma(x)$.

Assumption A. We require that there exist a constant $\mu_0 > 0$ such that $[\Sigma(x)] > \mu_0$ for all $x \in \Omega$.

Assumption A′. Ω and $\partial_1\Omega$ are the images under a bi-Lipschitz mapping of some Ω' and $\partial_1\Omega'$ which satisfy the assumption A.

We denote, for any $y \in \mathbf{R}^n$, the ball of centre y and radius ρ by $I(y,\rho)$ and by $S(y,\rho) = \partial I(y,\rho)$ the sphere of centre y and radius ρ. We set

$$\Omega(y,\rho) = \Omega \cap I(y,\rho), \ \bar{\Omega}(y,\rho) = \bar{\Omega} \cap I(y,\rho).$$

It will be convenient to introduce the following

DEFINITION. Let A be a bounded open set in $\mathbf{R}^n$ and $\beta > 0$ be a constant. $\mathscr{F}(\beta, A)$ denotes the family of all subsets B of $\bar{A}$ such that the following inequality holds for all $u \in C^1(\bar{A})$ vanishing on B

$$\| u \|_{q^*,A} \leqq \beta \| u_x \|_{q,A}$$

where

$$1/q^* = 1/q - 1/n \text{ for all } 1 < q \leqq n.$$

We shall require that Ω satisfies a mild assumption of admissibility described below.

Assumption B. (0) For all $y \in \partial\Omega$ we have

$$\liminf_{\rho \to 0} \frac{|\Omega(y,\rho)|}{|I(y,\rho)|} > 0.$$

There exist a constant $\beta > 0$ and, for all $y \in \partial\Omega$, a $\bar{\rho}(y) > 0$ such that

(i) for all $y \in \overline{\partial_1\Omega}$ and $0 < \rho < \bar{\rho}(y)$,

$$\Omega \cap S(y,\rho) \in \mathscr{F}(\beta, \Omega(y,\rho));$$

(ii) for all $y \in \partial_2\Omega$ and $0 < \rho < \bar{\rho}(y)$, every subset E of $\Omega(y,\rho)$ such that $|E| > \frac{1}{2}|\Omega(y,\rho)|$ belongs to the family $\mathscr{F}(\beta, \Omega(y,\rho))$.

Sufficient conditions for the validity of the Assumption B can be found in Stampacchia [19].

We consider on $\bar{\Omega}$ a linear uniformly elliptic second order differential operator of the form

$$Au = -\frac{\partial}{\partial x_k}(a_{jk}(x)u_{x_j}) \tag{1.6}$$

where the coefficients a_{jk} are bounded measurable functions defined on $\bar{\Omega}$ satisfying

$$m|\xi|^2 \leqq a_{jk}(x)\xi_j\xi_k \leqq M|\xi|^2, \text{ for all } \xi \in \mathbf{R}^n \text{ and a.e in } \bar{\Omega}, \tag{1.7}$$

with some constant of ellipticity $m > 0$. We shall write

$$a(u,v) = \int_\Omega a_{jk}(x)u_{x_j}(x)v_{x_k}(x)dx. \tag{1.8}$$

Then it is clear that there exists a constant $C > 0$ such that

$$|a(u,v)| \leqq C \|u\|_V \|v\|_V, \text{ for all } u,v \in V, \tag{1.9}$$

and hence A maps V continuously into its dual space V'.

On the other hand, under the Assumption A or A′ made on Ω and $\partial_1\Omega$, it follows, by the uniform ellipticity of A, that $a(u,v)$ is coercive on V; that is, there exists a constant $c > 0$ such that

$$a(u,u) \geqq c\|u\|_V^2, \text{ for all } u \in V. \tag{1.10}$$

Now suppose that ψ (referred to as the obstacle) is a given function in $H^1(\Omega)$ such that $\psi \leqq 0$ on $\partial_1\Omega$ (in the sense defined earlier).

Let us set

$$\mathbf{K} = \{u \in V;\ u \geqq \psi \text{ in } \Omega\} = \{u \in V;\ u - \psi \geqq 0 \text{ in } \Omega\}. \tag{1.11}$$

It is clear that $\mathbf{K}$ is a closed convex subset of V.

Let $T \in V'$ be given. We shall be concerned with the variational inequality

$$u \in \mathbf{K};\ a(u, v-u) \geqq \langle T, v-u\rangle, \text{ for all } v \in \mathbf{K}, \tag{1.12}$$

where $\langle\, ,\rangle$ denotes the pairing between V and V'.

Since the bilinear form $a(u,v)$ is continuous (in the sense that it satisfies (1.9)) and coercive, it follows by a well-known result on existence of solutions of variational inequalities (see, for instance, Lions and Stampacchia [11]), that there exists a unique solution $u \in \mathbf{K}$ of the variational inequality (1.12).

For the sake of simplicity all the calculations will be carried out assuming that $n > 2$. However, all the results hold also for $n = 2$ with minor changes.

We note that, when $\partial_2\Omega$ is Lipschitz, the functionals of the form

$$\langle T, v\rangle = \int_\Omega (f_0 v + f_j v_{x_j})\,dx + \int_{\partial_2\Omega} gv\,d\sigma, \text{ for all } v \in V, \tag{1.13}$$

belong to V' provided that

$$\begin{cases} f_0 \in L^r(\Omega),\ r \geqq 2n/(n+2);\ f_j \in L^p(\Omega),\ p \geqq 2, \text{ for } j = 1, \cdots, n; \\ g \in L^q(\partial_2\Omega),\ q \geqq 2(n-1)/n \end{cases} \tag{1.14}$$

($d\sigma$ denotes the $(n-1)$-dimensional volume element on $\partial_2\Omega$). This follows immediately on applying Hölder's inequality together with the Sobolev inequality for v in V and the fact that v in V admits a trace on $\partial_2\Omega$ which belongs to $L^s(\partial_2\Omega)$.

Let us remark that when $\partial_1\Omega$ does not satisfy (1.4), the coerciveness of $a(u,v)$ fails and the problem becomes only semi-coercive in the sense of Lions and Stampacchia, i.e., $a(v,v) \geqq c\| v_x \|^2_{2\,\Omega}$ and $\| v_x \|_{2,\Omega}$ is no longer a norm on $V = H^1(\Omega)$. A sufficient condition in order that the solution still exist is that

$$\int_\Omega f_0\,dx + \int_{\partial_2\Omega} g\,d\sigma < 0.$$

For details we refer to Lions and Stampacchia [11, § 6].

In the remainder of the paper, we shall be interested in the properties of the solution of the variational inequality (1.12) and in the possibility of regularizing it.

2. Hölder continuity of the solution

This section is concerned with first theorems of regularity for solutions of the variational inequality (1.12). In the first part, we prove certain a priori global estimates which give the boundedness of the solution. We then derive local a priori estimates which we use to prove that the solution is Hölder continuous up to the boundary provided that the functional T on the right hand side of the variational inequality (1.12) is defined by functions $f_0, f_1, \cdots, f_n$ and g belonging to suitable L^p-spaces, and that the obstacle ψ is in some $H^{1,p}(\Omega)$.

We begin with the global estimates. The method of proof is analogous to that used in Stampacchia [21], Murthy and Stampacchia [15] and Da Veiga [23] and so we limit ourselves to indicate only the salient points.

We assume that $\partial_2\Omega$ admits (locally) a Lipschitz representation and that, besides the previous assumptions, the obstacle ψ is bounded in $\bar{\Omega}$ with $\psi \leqq 0$ on $\partial_1\Omega$.

Let u be the solution of the variational inequality (1.12) and $p \geqq 2$.

Let $k_0 = \max(\max_{\bar{\Omega}}\psi, 0)$. For any real number $k \geqq k_0$ let $v = \min(u, k)$ which is clearly in the convex set $\mathbf{K}$. If $A(k)$ denotes the set $\{x \in \bar{\Omega};\, u(x) > k\}$ then, since $v - u$ vanishes in $\bar{\Omega} - A(k)$, we obtain on substituting this v in the variational inequality (1.12)

$$\int_{A(k)} a_{jl}u_{x_j}u_{x_l}\,dx \leqq \int_{A(k)} (f_0(u-k) + f_j u_{x_j})dx + \int_{A(k)\cap\partial_2\Omega} g(u-k)\,d\sigma.$$

The right hand side here can be estimated from above using Hölder's inequality together with Sobolev's inequality. The left hand side can be estimated from below, first by using the ellipticity of A and then on applying Sobolev's inequality in both the following forms.

There exists a constant $C > 0$ such that

$$\|\zeta\|_{2^*\Omega} \leqq C\,\|\zeta\|_V, \text{ for } \zeta \in V \text{ with } 1/2^* = \tfrac{1}{2} - 1/n$$

and

$$\|\zeta\|_{s,\partial_2\Omega} \leqq C\,\|\zeta\|_V, \text{ for } \zeta \in V \text{ with } s = 2(n-1)(n-2)^{-1}.$$

Setting

$$\mu(k) = |A(k)| + [A(k) \cap \partial_2\Omega]^{2^*/s}$$

we obtain

$$\mu(h) \leqq C(h-k)^{-2^*}\,\mu(k)^{\beta} \text{ for all } h > k > k_0$$

where

$$C = \{\|f_0\|^2_{p/2,\Omega} + \sum_j \|f_j\|^2_{p,\Omega} + \|g\|^2_{q,\partial_2\Omega}\}^{2^*/2}$$

and

$$\beta = \min\{(1-2/p)2^*/2,\ (1-1/s-1/q)s\}.$$

We observe that if $p > n$ and $q > n-1$, then $\beta > 1$. Using an algebraic lemma which can be found in [21] (for the non-negative function μ defined on $h > k_0$), we find that

$$\mu(k_0 + d) = 0$$

where

$$d^{2^*} \leqq C\,[\mu(k_0)]^{\beta-1}.$$

This argument proves the following

THEOREM 2.1. *If $u \in \mathbf{K}$ is the solution of the variational inequality (1.12) where*

$$\psi \in L^\infty(\Omega) \cap H^1(\Omega) \text{ with } \psi \leqq 0 \text{ on } \partial_1\Omega,$$

$$f_0 \in L^{p/2}(\Omega),\ f_j \in L^p(\Omega) \text{ for } j = 1,\cdots,n \text{ with } p > n,$$

$$\text{and } g \in L^q(\partial_2\Omega) \text{ with } q > n-1,$$

then we have

$$\psi(x) \leqq u(x) \leqq \max\,(\max_{\bar\Omega}\psi, 0) + \left\{\|f_0\|^2_{p/2\,\Omega} + \sum_j \|f_j\|^2_{p\,\Omega} + \|g\|^2_{q\,\partial_2\Omega}\right\}^{\frac{1}{2}}$$

almost everywhere in $\bar\Omega$.

The same tools can also be used to prove

THEOREM 2.2 *If $u \in \mathbf{K}$ is a solution of the variational inequality with $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$, $f_0 \in L^{p/2}(\Omega)$, $f_j \in L^p(\Omega)$, $g \in L^q(\partial_2\Omega)$ for $j = 1,\cdots,n$ with $2 \leqq p < n$ and $2(n-1)/n \leqq q < n-1$, then $u \in \mathbf{K} \cap L^{p^*}(\Omega)$ $(1/p^* = 1/p - 1/n)$.*

We now proceed to prove the Hölder continuity of the solution u of (1.12) up to the boundary. The proof is based on a method essentially due to De Giorgi [4] for uniformly elliptic equations. This was later extended by Ladyzenkaja and Ural'-tceva [6], Morrey [13] and Stampacchia [19] to uniformly elliptic linear and non-linear equations and boundary value problems. Further extensions to degenerate elliptic equations are given by Murthy and Stampacchia [15] and to some types of variational inequalities by Da Veiga [23].

The method consists essentially of taking, for the test function v in the variational inequality (1.12), a function obtained by suitably truncating u and then localizing it.

We assume in the rest of this section that the assumptions A (or A′) and B hold and that $\partial_2\Omega$ admits (locally) a Lipschitz representation. Let

$$\text{(2.1)} \qquad \begin{cases} f_0 \in L^{p/2}(\Omega),\ f_j \in L^p(\Omega)\ (j = 1,\cdots,n) \text{ with } p > n; \\ g \in L^q(\partial_2\Omega) \text{ with } q > n-1. \end{cases}$$

Moreover, the obstacle ψ is assumed to be in $H^{1,p}(\Omega) \cap C^{0,\gamma}(\bar\Omega)$, for some $0 < \gamma < 1$ and the same p, with $\psi \leqq 0$ on $\partial_1\Omega$.

We, first of all, obtain a local a priori estimate for u in the following.

Let $u \in \mathbf{K}$ be the solution of the variational inequality (1.12) and $x_0 \in \bar\Omega$. If

$x_0 \in \partial\Omega$, then there exists a $\rho_0(x_0) = \rho_0 > 0$ given by the Assumption B. If $x_0 \in \Omega$, there exists a $\rho_0(x_0) > 0$ such that $I(x_0, \rho_0(x_0)) \subset \Omega$.

In the following, we distinguish two cases; namely, (i) $x_0 \in \overline{\partial_1\Omega}$ and (ii) $x_0 \in \Omega \cup \partial_2\Omega$. If $x_0 \in \Omega \cup \partial_2\Omega$ we may, without loss of generality, suppose that $\rho_0(x_0) = \rho_0$ is such that $I(x_0, \rho_0) \cap \partial_1\Omega = \varnothing$. (We can always assume that $0 < \rho_0 < 1$).

Let ρ and R be two real numbers such that $0 < \rho < R < \rho_0(x_0)$ and let $\zeta \in C_0^1(I(x_0, \rho_0))$ be a function such that

$$(2.2) \qquad \begin{cases} 0 \leqq \zeta(x) \leqq 1, \ \zeta(x) = \begin{cases} 1 \text{ in } I(x_0, \rho) \\ 0 \text{ outside } I(x_0, R) \end{cases} \text{ and} \\ |\zeta_x(x)| \leqq \text{const. } (R-\rho)^{-1}. \end{cases}$$

We also set, eventually reducing ρ_0,

$$(2.3) \qquad \begin{aligned} k_0 &= k_0(\rho_0) = \max(\sup_{\Omega(x_0,\rho_0)} (-\psi), 0); \\ l_0 &= l_0(\rho_0) = \min(\inf_{\Omega(x_0,\rho_0)} (-\psi), 0) \geqq 0. \end{aligned}$$

For any real number $k \geqq 0$ we consider the two functions $v, v' \in H^1(\Omega)$ defined by the relations

$$(2.4) \qquad v = u - \zeta^2 \max(u - \psi - k, 0), \ v' = u - \zeta^2 \min(u - \psi - k, 0).$$

It can now be easily verified that we have

(i) if $x_0 \in \overline{\partial_1\Omega}$, then $v \in \mathbf{K}$ for all $k \geqq k_0$ and $v' \in \mathbf{K}$ for all $0 \leqq k \leqq l_0$,

(ii) if $x_0 \in \Omega \cup \partial_2\Omega$, then $v, v' \in \mathbf{K}$ for any $k \geqq 0$.

Since x_0 is fixed, we say in the following that a number k is admissible if the function v (resp. v') given by (2.4) belongs to $\mathbf{K}$.

We denote, for any $0 < t < \rho_0$, by $A(k, t)$ and $B(k, t)$ the sets defined respectively by

$$(2.5) \qquad \begin{cases} A(k,t) = \{x \in \bar{\Omega}(x_0, t);\ u(x) \geqq \psi(x) + k\} \\ B(k,t) = \{x \in \bar{\Omega}(x_0, t);\ u(x) \leqq \psi(x) + k\}. \end{cases}$$

Then we have

$$v - u = \begin{cases} -\zeta^2(u - \psi - k) \text{ in } A(k,R); \\ 0 \quad \text{otherwise} \end{cases} \qquad v' - u = \begin{cases} -\zeta^2(u - \psi - k) \text{ in } B(k,R) \\ 0 \quad \text{otherwise.} \end{cases}$$

If k is admissible, then on substituting the function $v \in \mathbf{K}$ in the variational inequality (1.12) and adding the term

$$\int_{A(k,R)} a_{ij} \psi_{x_i} [\zeta^2 (u - \psi - k)]_{x_j} dx$$

to both sides, we find that

$$\int_{A(k,R)} a_{ij} \zeta^2 (u - \psi)_{x_i} (u - \psi)_{x_j} dx$$

$$\leqq \int_{A(k,R)} \{(f_j - a_{ij}\psi_{x_i})\zeta^2(u-\psi)_{x_j} + \zeta(f_0\zeta + 2\zeta_{x_j}(f_j - a_{ij}\psi_{x_i}))(u-\psi-k)\} dx \tag{2.6}$$

$$+ \int_{A(k,R)\cap\partial_2\Omega} g\zeta^2(u - \psi - k)\, d\sigma.$$

The left hand side here can immediately be estimated from below by using the ellipticity to get

$$m\int_{A(k,R)} \zeta^2 |(u - \psi - k)_x|^2 dx \leqq \int_{A(k,R)} a_{ij}\zeta^2 (u - \psi)_{x_i}(u - \psi)_{x_j} dx.$$

In order to estimate the right hand side of (2.6), let us assume that $A(k, \rho_0) \in \mathscr{F}(\beta, \Omega(x_0, \rho_0))$ so that $\zeta(u - \psi - k) \in L^{2^*}(A(k, R))$ and its norm is majorized by $\beta \| [\zeta(u - \psi - k)]_x \|_{2,A(k,R)}$.

Moreover, since $\partial_2\Omega$ has a (local) Lipschitz representation,

$$\zeta(u - \psi - k) \in L^s(A(k, R) \cap \partial_2\Omega)$$

and its norm is again majorized by $\| [\zeta(u - \psi - k)]_x \|_{2,A(k,R)}$.

Hence, on applying Hölder's inequality to each term on the right side of (2.6), we obtain the estimate

$$\int_{A(k,R)} \zeta^2 |(u - \psi - k)_x|^2 dx \leqq \text{const.} \left\{ \int_{A(k,R)} |\zeta_x|^2 |u - \psi - k|^2 dx \right.$$

$$+ \int_{A(k,R)} \zeta^2 [|f_0|^2 |A(k,R)|^{2/n} + \Sigma |f_j|^2 + |\psi_x|^2]\, dx$$

$$\left. + \left(\int_{A(k,R)\cap\partial_2\Omega} |\zeta g|^{s'} d\sigma \right)^{2/s'} \right\}$$

where $s' = 2(n - 1)/n$.

Since $\zeta = 1$ in $\bar{\Omega}(x_0, \rho) \subset I(x_0, \rho)$ and taking into account (2.2) and Assumption B, we have thus proved

THEOREM 2.3 *If $u \in \mathbf{K}$ is a solution of the variational inequality* (1.12), *then we have*

$$\text{(a)} \quad \int_{A(k,\rho)} |(u-\psi-k)_x|^2\, dx \leqq \text{const.} \Big\{ (R-\rho)^{-2} \int_{A(k,R)} |u-\psi-k|^2\, dx + \int_{A(k,R)} [|f_0|^2 |A(k,R)|^{2/n} + \sum_j |f_j|^2 + |\psi_x|^2]\, dx + \Big(\int_{A(k,R)\cap \partial_2\Omega} |g|^{s'} d\sigma \Big)^{2/s'} \Big\}$$

(i) *for all $k \geqq k_0 \geqq 0$ if $x_0 \in \overline{\partial_1\Omega}$ and*

(ii) *for all $k \geqq 0$ such that $|A(k,\rho_0)| \leqq \frac{1}{2}|\Omega(x_0,\rho_0)|$ if $x_0 \in \Omega \cup \partial_2\Omega$;*

(b) *The same estimate holds with $B(k,\rho)$ and $B(k,R)$ in place of $A(k,\rho)$ and $A(k,R)$*

(i) *for all $0 \leqq k \leqq l_0$ when $x_0 \in \overline{\partial_1\Omega}$ and*

(ii) *for all $k \geqq 0$ such that $|B(k,\rho_0)| \leqq \frac{1}{2}|\Omega(x_0,\rho_0)|$ when $x_0 \in \Omega \cup \partial_2\Omega$.*

Part (b) follows in the same way by taking $v' \in \mathbf{K}$ in place of v in the above proof.

We note that for $x_0 \in \Omega \cup \partial_2\Omega$ either condition (a) (ii) or condition (b) (ii) of Theorem 2.3 always holds.

We recall two lemmas which will be used to derive a local boundedness estimate from the above theorem.

LEMMA 2.4 *Let Ω satisfy the Assumption B. Then there exists a constant $c_0 > 0$ such that for every $v \in V$, $x_0 \in \bar{\Omega}$ and $0 < \rho < \rho_0(x_0)$, we have*

$$\int_{A(k,\rho)} |v-k|^2 dx \leqq c_0 \int_{A(k,\rho)} |v_x|^2 dx \cdot |A(k,\rho)|^{2/n},$$

$$|A(h,\rho)|^{(n-2)/n} \leqq c_0 (h-k)^{-2} \int_{A(k,\rho)-A(h,\rho)} |v_x|^2 dx$$

where $h > k$

(i) *for every $k \geqq 0$ if $x_0 \in \overline{\partial_1\Omega}$ and*

(ii) *for every $k \geqq 0$ such that*

$$|A(k,\rho)| \leqq \tfrac{1}{2}|\Omega(x_0,\rho)|$$

if $x_0 \in \Omega \cup \partial_2\Omega$. Similar inequalities hold with $A(k,\rho)$ and $A(h,\rho)$ replaced by $B(k,\rho)$ and $B(h,\rho)$ where $0 \leqq h \leqq k$,

(i) *for all $k \geqq 0$ if $x_0 \in \overline{\partial_1\Omega}$ and*

(ii) *for all* $k \geqq 0$ *such that* $|B(k,\rho)| \leqq \frac{1}{2}|\Omega(x_0,\rho)|$ *if* $x_0 \in \Omega \cup \partial_2\Omega$.

LEMMA 2.5 *Under the same assumptions as in the Lemma 2.4, there exists a constant* $c_1 > 0$ *such that*

$$|A(h,\rho)|^{(2n-2)/n} \leqq c_1(h-k)^{-2}\int_{A(k,\rho)} |v_x|^2 dx \cdot \{|A(k,\rho)| - |A(h,\rho)|\}$$

with the same restrictions for k. A similar assertion holds also for $B(h,\rho)$ *and* $B(k,\rho)$ *in place of* $A(h,\rho)$ *and* $A(k,\rho)$.

These lemmas are consequences of the fact that if a function v vanishes in a set $E \subset \Omega(x_0,\rho)$ with $|E| > \frac{1}{2}|\Omega(x_0,\rho)|$ (which is the case here for $E = A(k_0,\rho)$ or $B(k_0,\rho)$), then $E \in \mathscr{F}(\beta, \Omega(x_0,\rho))$ by Assumption B and so we can write Sobolev's inequality. For details of the proof, refer to Stampacchia [19].

For simplicity of writing we set

$$\begin{aligned} G(k,R) &= \int_{A(k,R)} \{|f_0|^2 |A(k,R)|^{2/n} + \sum_j |f_j|^2 + |\psi_x|^2\} dx \\ &\quad + \left(\int_{A(k,R)\cap\partial_2\Omega} |g|^{s'} d\sigma\right)^{2/s'} . \end{aligned} \tag{2.7}$$

The two lemmas above, together with the theorem 2.3, imply the

THEOREM 2.6 *Suppose u is a solution of the variational inequality* (1.12) *and* Ω *satisfies A (or A') and B.*

(a) *If* $x_0 \in \overline{\partial_1\Omega}$ *and* $0 < \rho < R < \rho_0(x_0)$ *then, for all* $h > k \geqq k_0$, *we have the following estimates*:

(i) $$\int_{A(h,\rho)} (u-\psi-h)^2 dx \leqq \text{const.} \left\{(R-\rho)^{-2}\int_{A(k,R)} (u-\psi-k)^2 dx + G(k,R)\right\} \times |A(k,R)|^{2/n};$$

(ii) $$|A(h,\rho)|^{(n-2)/n} \leqq \text{const.}\ (h-k)^{-2}\left\{(R-\rho)^{-2}\int_{A(k,R)} (u-\psi-k)^2 dx + G(k,R)\right\};$$

(iii) $$|A(h,\rho)|^{(2n-2)/n} \leqq \text{const.}\ (h-k)^{-2}\left\{(R-\rho)^{-2}\int_{A(k,R)} (u-\psi-k)^2 dx + G(k,R)\right\} \times \{|A(k,\rho)| - |A(h,\rho)|\}.$$

(b) *The same estimates hold also for* $x_0 \in \Omega \cup \partial_2\Omega$ *and for all* $h > k \geqq 0$ *provided that*

$$|A(k,\rho_0)| \leqq \tfrac{1}{2}|\Omega(x_0,\rho_0)|.$$

(c) *Similar estimates hold with* $B(h,\rho)$ *and* $B(k,R)$ *in place of* $A(h,\rho)$ *and* $A(k,R)$ *if*

(i) $x_0 \in \overline{\partial_1 \Omega}$ *and* $h < k \leqq l_0$ *or*

(ii) $x_0 \in \Omega \cup \partial_2 \Omega$ *and* $0 \leqq h < k$ *provided that*

$$|B(k,\rho_0)| \leqq \tfrac{1}{2}|\Omega(x_0,\rho)| \ .$$

Since the data $f_0, f_1, \cdots, f_n$ and g satisfy (2.1) and since $\psi \in H^{1,p}(\Omega)$ so that $|\psi_x| \in L^p(\Omega)$, we can estimate $G(k,R)$ by Hölder's inequality and we find

$$G(k,R) \leqq \|f_0\|^2_{p/2}|A(k,R)|^{1-4/p+2/n} + \sum_j (\|f_j\|^2_p + \|\psi_x\|^2_p)|A(k,R)|^{1-2/p+2/n} \tag{2.8}$$
$$+ \|g\|^{2/s'}_{q,\partial_2\Omega}[A(k,R)\cap\partial_2\Omega]^{2/s'-2/q}$$

where the norms of $f_0, f_1, \cdots, f_n$ and ψ_x are taken over Ω.

Let r be a real number such that $0 < r/2 \leqq \rho < R \leqq 2r < \rho_0(x_0)$. Then the estimates of Theorem 2.6 evidently hold with ρ replaced by $r/2$ and R by $2r$. If we now set, for all $0 < \rho < \rho_0$ and $h \geqq 0$,

$$a(h,\rho) = \int_{A(h,\rho)} (u - \psi - h)^2 dx,$$
$$\mu(h,\rho) = |A(h,\rho)| + [A(h,\rho)\cap\partial_2\Omega]^{2*/s},$$

then an iterative procedure employed by Stampacchia in [19] and the estimate (2.8) for $G(k,R)$ show that

$$\mu(k_0 + d/2, r/2) = 0 \tag{2.9}$$

where $d > 0$ is any number such that

$$d^2 \geqq \text{const. } r^{-n}\int_{A(k_0,2r)} (u - \psi - k_0)^2 dx \tag{2.10}$$
$$+ \text{ const.}\left\{\left[\|f_0\|^2_{p/2} + \sum \|f_j\|^2_p + \|\psi_x\|^2_p\right] r^{2(1-n/p)} + \|g\|^{2/s'}_{q,\partial_2\Omega} r^{2(1-(n-1)/q)}\right\}$$

where the constants depend only on Ω, n, p, q but are independent of r.

In other words, we have the following estimate for the local boundedness of u·

THEOREM 2.7 *Suppose that* Ω *satisfies the assumption* A (*or* A') *and* B, *and* $u \in \mathbf{K}$ *be a solution of the variational inequality with* $f_0, \cdots, f_n$ *and* g *satisfying* (2.1).

(a) *If* $x_0 \in \overline{\partial_1 \Omega}$ *and* $0 < 2r < \rho_0(x_0)$, *then there exists a constant* $C > 0$ *depending only on* Ω, n, p, q *but independent of* r *such that*

$$(2.11)\quad \sup_{\bar{\Omega}(x_0,r/2)} (u-\psi)(x) \leqq k_0 + C\left\{r^{-n}\int_{A(k_0,2r)} |u-\psi-k_0|^2 dx + Gr^{2\alpha}\right\}^{\frac{1}{2}}$$

where

$$(2.12)\quad \begin{cases} 0 < \alpha = \min(1-n/p,\, 1-(n-1)/q) < 1 \text{ and} \\ G = \|f_0\|^2_{p/2} + \sum_j \|f_j\|^2_p + \|\psi_x\|^2_p + \|g\|^{2/s'}_{q,\partial_2\Omega}. \end{cases}$$

(b) *The estimate* (2.11) *remains valid for* $x_0 \in \Omega \cup \partial_2\Omega$ *with any* $k_0 \geqq 0$ *provided that, for all* $0 < \rho < \rho_0(x_0)$,

$$|A(k_0,\rho)| \leqq \tfrac{1}{2}|\Omega(x_0,\rho)|.$$

(c) *A similar estimate from below holds for* $\inf_{\bar{\Omega}(x_0,r/2)}(u-\psi)(x)$ *with* l_0 *in place of* k_0 *if* $x_0 \in \overline{\partial_1\Omega}$.

We also observe that, since $u \geqq \psi$, we always have

$$\inf_{\bar{\Omega}(x_0,r/2)} (u-\psi)(x) \geqq 0 \text{ for } x_0 \in \Omega \cup \partial_2\Omega.$$

We are now in a position to prove that u is Hölder continuous up to the boundary. For this purpose we define, for $x_0 \in \bar{\Omega}$ and $0 < \rho < \rho_0(x_0)$,

$$(2.13)\quad \begin{cases} M(\rho) = \sup_{\bar{\Omega}(x_0,\rho)} (u-\psi)(x), \; m(\rho) = \inf_{\bar{\Omega}(x_0,\rho)} (u-\psi)(x) \\ \text{and } \omega(\rho) = M(\rho) - m(\rho). \end{cases}$$

Since $u \geqq \psi$ in Ω, it follows that $M(\rho) \geqq m(\rho) \geqq 0$ and, moreover, $M(\rho)$ (and hence also $m(\rho)$) is finite by Theorem 2.1. For an integer $N \geqq 0$ and $0 < 2r < \rho_0(x_0)$, we set

$$(2.14)\quad K_N = M(2r) - 2^{-(N+1)}\omega(2r),\; L_N = m(2r) + 2^{-(N+1)}\omega(2r).$$

Then K_N is an increasing sequence tending to $K_\infty = M(2r)$ while L_N is decreasing tending to $L_\infty = m(2r)$. It is clear that $K_0 = L_0 = \frac{1}{2}(M(2r) + m(2r))$ and $K_N \geqq 0$, $L_N \geqq 0$.

REMARK. 1. If $x_0 \in \Omega \cup \partial_2\Omega$, then every K_N (resp. every L_N) is admissible for defining the test function v (resp. v') by (2.4).

REMARK. 2. If $x_0 \in \overline{\partial_1\Omega}$, every $K_N \geqq k_0 = \max(\sup_{\bar{\Omega}(x_0,\rho)}(-\psi), 0)$ [resp. every $L_N \leqq l_0 = \min(\inf_{\bar{\Omega}(x_0,\rho)}(-\psi), 0) \geqq 0$] is admissible for defining the test function v (resp. v') by (2.4). Thus there are three possible cases to be considered.

(α) There is an integer $N_0 \geqq 0$ such that $K_N \geqq k_0$ for $N \geqq N_0$.

(β) There is an integer $N_0 \geqq 0$ such that $L_N \leqq l_0$ for $N \geqq N_0$.

(γ) For every integer N we have $l_0 \leqq L_N \leqq L_0 = K_0 \leqq K_N \leqq k_0$.

REMARK 3. Suppose some K_N (and hence also all the succeeding ones) is admissible for defining v in the sense that (α) holds. Then using Theorem 2.6 (iii), it is possible to prove that

$$|A(K_N, 2r)| \to 0 \text{ as } K_N \to K_\infty = M(2r).$$

Thus if some K_N is admissible for defining v, then for sufficiently large N we may assume that

$$|A(K_N, 2r)| < \tfrac{1}{2}|\Omega(x_0, 2r)|$$

and

$$Cr^{-n}|A(K_N, 2r)| < \tfrac{1}{4},$$

where C is the constant of Theorem 2.7. A similar remark applies also for $B(L_N, 2r)$ if some L_N is admissible for defining v' in the sense that (β) holds. Then we are in a position to prove.

THEOREM 2.8 *Under the assumptions of the Theorem 2.7, there exist two constants* $0 < \eta < 1$ *and* $H > 0$ *(*H *depending only on the norms of* $f_0, f_1, \cdots, f_n, g, \Omega, n, p, q$ *but independent of* r*) such that*

$$\omega(r/2) \leqq \eta\omega(4r) + Hr^\alpha$$

($0 < \alpha < 1$ *is defined by* $\alpha = \min(1 - n/p,\ 1 - (n-1)/q)$).

PROOF. Let $x_0 \in \Omega \cup \partial_2\Omega$ and suppose that some K_N is admissible for defining v. Then by Remarks 1. and 3. above, there is an integer N_0 such that, for $N \geqq N_0$, we have

$$|A(K_N, 2r)| \leqq \tfrac{1}{2}|\Omega(x_0, 2r)| \text{ and } Cr^{-n}|A(K_N, 2r)| < \tfrac{1}{4}.$$

We can then apply Theorem 2.7 (a) with k_0 replaced by K_N and we find

$$\begin{aligned} M(r/2) &\leqq K_N + C^{\frac{1}{2}}[(M(2r) - K_N)^2 r^{-n}|A(K_N, 2r)| + Gr^{2\alpha}]^{\frac{1}{2}} \\ &\leqq K_N + \tfrac{1}{2}(M(2r) - K_N) + C^{\frac{1}{2}}G^{\frac{1}{2}}r^\alpha \\ &= M(2r) - 2^{-(N+2)}\omega(2r) + C^{\frac{1}{2}}G^{\frac{1}{2}}r^\alpha. \end{aligned}$$

Now since $m(r/2) \geqq m(2r)$, we get

$$\omega(r/2) \leqq (1 - 2^{-(N+2)})\omega(2r) + \tfrac{1}{2}G^{\frac{1}{2}}r^\alpha,$$

which proves the assertion in this case.

Now let $x_0 \in \overline{\partial_1 \Omega}$. In case ($\alpha$) of Remark 2., the above proof again works. Consider case (β) of Remark 2. Then, for all $N \geqq N_0$, we have

$$|B(L_N, 2r)| \leqq \tfrac{1}{2}|\Omega(x_0, 2r)| \text{ and } Cr^{-n}|B(L_N, 2r)| < \tfrac{1}{4}$$

and we can apply Theorem 2.7 (c) to obtain, as before,

$$\begin{aligned} m(r/2) &\geqq L_N - C^{\frac{1}{2}}\left[r^{-n}\int_{B(L_N,2r)}(u - \psi - L_N)^2 dx + Gr^{2\alpha}\right]^{\frac{1}{2}} \\ &\geqq L_N - \tfrac{1}{2}(L_N - m(2r)) - C^{\frac{1}{2}}G^{\frac{1}{2}}r^{\alpha} \end{aligned}$$

because

$$\begin{aligned} C^{\frac{1}{2}}r^{-n}\int_{B(L_N\ 2r)}(u - \psi - L_N)^2 dx &\leqq C^{\frac{1}{2}}r^{-n}(L_N - m(2r))^2\,|B(L_N, 2r)| \\ &\leqq (\tfrac{1}{4})(L_N - m(2r))^2. \end{aligned}$$

From this, exactly as above, we have

$$\begin{aligned} m(r/2) &\geqq L_N - \tfrac{1}{2} \times 2^{-(N+1)}\omega(2r) - \tfrac{1}{2}G^{\frac{1}{2}}r^{\alpha} \\ &= m(2r) + 2^{-(N+2)}\omega(2r) - \tfrac{1}{2}G^{\frac{1}{2}}r^{\alpha}. \end{aligned}$$

Since $M(r/2) \leqq M(2r)$, we thus get

$$\omega(r/2) = M(r/2) - m(r/2) \leqq \omega(2r)(1 - 2^{-(N+2)}) + \tfrac{1}{2}G^{\frac{1}{2}}r^{\alpha}$$

which again proves the required assertion.

If $x_0 \in \Omega \cup \partial_2\Omega$ and some L_N is admissible for the definition of v', the same arguments apply.

Finally in case (γ) of Remark 2, we have

$$l_0 \leqq L_N \leqq K_N \leqq k_0 \text{ for all } N.$$

Hence, letting $N \to +\infty$, we find that

$$l_0 \leqq M(2r) \leqq k_0 \text{ and } l_0 \leqq m(2r) \leqq k_0$$

and hence

$$l_0 - k_0 \leqq \omega(2r) \leqq k_0 - l_0.$$

Since $k_0 - l_0 = \sup_{\overline{\Omega}(x_0,2r)}(-\psi) - \inf_{\overline{\Omega}(x_0,2r)}(-\psi) = -\inf_{\overline{\Omega}(x_0,2r)}\psi + \sup_{\overline{\Omega}(x_0,2r)}\psi$ $= \operatorname{osc}_{\overline{\Omega}(x_0,2r)}\psi$, we have

$$\omega(r) = O(\operatorname*{osc}_{\overline{\Omega}(x_0,2r)} \psi).$$

This completes the proof of the theorem.

Now by a standard technique it follows that there exist constants $C > 0$ and $0 < \lambda < 1$ $(\lambda < \alpha)$ such that

$$\omega(r) \leqq Cr^{\lambda} \text{ for all } 0 < r < \rho_0(x_0).$$

This leads us to the principal result of this section which can be obtained by covering $\bar{\Omega}$ with a finite number of sets of the form $\bar{\Omega}(x_0, \rho_0(x_0))$ satisfying our requirements.

THEOREM 2.9 *Let Ω satisfy the Assumptions A (or A′) and B. Suppose $a_{ij} \in L^{\infty}(\Omega)$ and $a(u,v)$ be V-coercive. If $\psi \in H^{1,p}(\Omega) \cap C^{0,\gamma}(\bar{\Omega})$, $f_0 \in L^{p/2}(\Omega)$, $f_j \in L^p(\Omega)$, $j = 1, \cdots, n$ and $g \in L^q(\partial_2\Omega)$, with $p > n$ and $q > n - 1$, then any solution $u \in \mathbf{K}$ of the variational inequality* (1.12) *belongs to $V \cap C^{o,\lambda}(\bar{\Omega})$ where $0 < \lambda < 1$ depends only on $\Omega, \partial\Omega, n, p, q$ and γ.*

3. Further regularity

In this section, we show that the solution of the variational inequality (1.12) with $f_j = 0$ for $j = 1, \cdots, n$ can be approximated in $C^{0,\alpha}(\bar{\Omega}) \cap V$ by solutions of certain quasi-linear mixed boundary value problems associated with the elliptic operator A. The techniques used are very similar to those used in [7].

The approximation procedure permits us to obtain further regularity of the solution of the variational inequality using the corresponding regularity for solutions of the mixed problems. The existence and regularity results for mixed boundary problems have been considered by several authors and one can refer to the articles of Peetre [16] and Shamir [17] for a detailed bibliography.

We begin by recalling some known facts concerning mixed boundary value problems for the linear elliptic operator A. Then we obtain an existence theorem for a quasi-linear mixed boundary value problem related to our variational inequality by using the standard method of monotone operators.

We always suppose that the Assumption A (or A′) holds and so $a(u,v)$ is V-coercive.

Assume that $\partial_2\Omega$ has a locally Lipschitz representation so that every element u in V has a trace on $\partial_2\Omega$ which belongs to $L^s(\partial_2\Omega)$ $(s = 2(n-1)/(n-2))$. Given two functions $f \in L^r(\Omega)$ (see (1.14)) and $g \in L^{s'}(\partial_2\Omega)$ $(1/s + 1/s' = 1)$ we consider the problem of finding a solution

$$u \in V;\ a(u,v) = \int_{\Omega} fv\,dx + \int_{\partial_2\Omega} gv\,d\sigma, \text{ for all } v \in V. \tag{3.1}$$

We have already seen that the right hand side of (3.1) defines a continuous linear functional on V and hence, by the V-coerciveness of the bilinear form $a(u,v)$, there exists a unique solution $u \in V$ of (3.1) (by the Lax-Milgram lemma).

A solution $u \in V$ of problem (3.1) can be interpreted as a "generalized" solution of the mixed boundary value problem (see Section 4).

$$Au = f \text{ in } \Omega;\; u = 0 \text{ on } \partial_1\Omega,\; (\partial u/\partial v) = g \text{ on } \partial_2\Omega. \tag{3.2}$$

REMARK. Under the Assumptions A (or A') and B, it is shown in Stampacchia [19] (see also, Murthy and Stampacchia [15]) that the solution $u \in V$ of the mixed boundary value problem (3.1) is Hölder continuous up to the boundary provided that $f \in L^p(\Omega)$ with $p > n/2$, $g \in L^q(\partial_2\Omega)$ with $q > n-1$, and that there exists a constant $C > 0$ and a number $0 < \lambda < 1$ such that

$$\| u \|_V + \| u \|_{C^{0,\lambda}(\bar{\Omega})} \leqq C\{\|f\|_{p,\Omega} + \| g \|_{q,\partial_2\Omega}\}. \tag{3.3}$$

We shall now consider a non-linear mixed boundary value problem.

Let $f \in L^p(\Omega)$ with $p > n/2$ and $g \in L^q(\partial_2\Omega)$ with $q > n-1$ be given. We shall make the following assumptions on the obstacle $\psi \in H^1(\Omega)$ with $\psi \leqq 0$ on $\partial_1\Omega$ and on the coefficients of A.

Assumption C. In the sense of distributions, $A\psi$ is a measure on Ω and $\partial\psi/\partial v$ is a measure on $\partial_2\Omega$ such that

$$\max(A\psi - f, 0) \in L^p(\Omega), p > n/2; \max((\partial\psi/\partial v) - g, 0) \in L^q(\partial_2\Omega), q > n-1.$$

The interior conormal $\partial\psi/\partial v$ is understood in the generalized sense. It is clear that, if the coefficients of A are in $C^1(\bar{\Omega})$ and $\psi \in C^2(\bar{\Omega})$ with $\psi \leqq 0$ on $\partial_1\Omega$, the Assumption C is satisfied.

We also note that, since $n \geqq 2$, $p > 2n/(n+2)$ and $q > s'$.

Let $\theta(t)$ be a non-increasing Lipschitz function on the real line $\mathbf{R}^1$ such that $0 \leqq \theta(t) \leqq 1$. Consider the non-linear mixed boundary value problem

$$\begin{cases} Au = \max(A\psi - f, 0)\theta(u - \psi) + f \text{ in } \Omega, \\ u = 0 \text{ on } \partial_1\Omega,\; \partial u/\partial v = \max((\partial\psi/\partial v) - g, 0)\theta(u - \psi) + g \text{ on } \partial_2\Omega. \end{cases} \tag{3.4}$$

The variational formulation of the mixed problem (3.4) can be defined by means of the quasi-linear form

$$\begin{aligned} b_\theta(u,v) \;=\; & a(u,v) - \int_\Omega \max(A\psi - f, 0)\theta(u - \psi)\, v\, dx \\ & - \int_{\partial_2\Omega} \max((\partial\psi/\partial v) - g, 0)\theta(u - \psi)\, v\, d\sigma, \text{ for } u, v \in V. \end{aligned}$$

The quasi-linear form $b_\theta(u,v)$ defines a non-linear operator $B \equiv B_\theta$ on V given by

$$b_\theta(u,v) = \langle Bu, v\rangle, \text{ for } u, v \in V. \tag{3.6}$$

The operator B is strictly monotone and continuous from V into V'. The strict monotonicity depends on the facts that $a(u,v)$ is V-coercive and that θ is non-increasing so that

$$[\theta(u-\psi) - \theta(v-\psi)](u-v) \leqq 0 \text{ for all } u, v \in V.$$

The continuity of B is a consequence of the continuity of the bilinear form $a(u,v)$ on V and of an easy estimate of the two terms involving the integrals in (3.5) obtained by applying Hölder's inequality together with Sobolev's inequalities.

Moreover the quasi-linear form $b_\theta(u,v)$ is coercive in the sense that

$$b_\theta(u,u)/\| u \|_V \to +\infty \text{ as } \| u \|_V \to +\infty.$$

We now observe that the problem (3.4) is equivalent to the problem of finding a solution of

$$u \in V;\ b_\theta(u,v) = \int_\Omega f v dx + \int_{\partial_2\Omega} g v d\sigma, \text{ for all } v \in V. \tag{3.7}$$

In view of this reduction, the general theory of monotone operators yields the following existence theorem.

THEOREM 3.1 *Suppose Ω satisfies Assumption A (or A') and $\partial_2\Omega$ is locally Lipschitz. If $f \in L^p(\Omega)$ with $p > 2n/(n+2)$ and $g \in L^q(\partial_2\Omega)$ with $q > s' = 2(n-1)/n$, and ψ and the coefficients of A satisfy Assumption C, then there exists a unique solution $u \in V$ of the non-linear mixed boundary value problem* (3.4).

As another consequence of the above reduction of the problem (3.4), we can derive, from estimate (3.3) of the Remark and the fact that $0 \leqq \theta(t) \leqq 1$, the following

THEOREM 3.2 *If Assumptions A (or A'), B and C hold with $p > n/2$ and $q > n-1$, then the solution u of the non-linear mixed problem* (3.4) *belongs to $C^{0,\lambda}(\bar\Omega) \cap V$ and we have*

$$\| u \|_V + \| u \|_{C^{0,\lambda}(\bar\Omega)} \leqq \text{const.} \{\| \max(A\psi - f, 0) \|_{p,\Omega} + \| f \|_{p,\Omega} + \| \max((\partial\psi/\partial\nu) - g, 0) \|_{q,\partial_2\Omega} + \| g \|_{q,\partial_2\Omega}\} \tag{3.8}$$

where the constant is independent of the function $\theta(t)$.

Hereafter we shall only be concerned with the variational inequality

$$u \in \mathbf{K};\ a(u, v-u) \geqq \int_\Omega f(v-u)\,dx + \int_{\partial_2\Omega} g(v-u)\,d\sigma \text{ for all } v \in K \tag{3.9}$$

where $f \in L^p(\Omega)$ and $g \in L^q(\partial_2\Omega)$ with $p > n/2$, $q > (n-1)$.

We shall show that the solution of the variational inequality (3.9) can be approximated from above as well as below by solutions of the mixed problems of the type (3.4) corresponding to sequences of functions $\theta(t)$ as described above. For similar results, see [7]. In the rest of this section, we shall use the notation

$$(3.10)\quad P(\psi,f) = \max(A\psi - f, 0) \text{ in } \Omega,\ Q(\psi,g) = \max((\partial\psi/\partial\nu) - g, 0) \text{ on } \partial_2\Omega.$$

We consider two sequences of functions of the type $\theta(t)$ defined as follows:

$$(3.11)\qquad \theta'_m(t) = \begin{cases} 1, & \text{for } t \leqq -1/m \\ -mt, & \text{for } -1/m \leqq t \leqq 0 \\ 0, & \text{for } t \geqq 0 \end{cases}$$

and

$$(3.12)\qquad \theta''_m(t) = \begin{cases} 1, & \text{for } t \leqq 0 \\ -mt + 1, & \text{for } 0 \leqq t \leqq 1/m \\ 0, & \text{for } t \geqq 1/m. \end{cases}$$

Then $\theta'_m(t)$ is a non-decreasing sequence of functions each of which is Lipschitz and non-increasing while $\theta''_m(t)$ is a non-increasing sequence of functions with the same properties. Both the sequences "converge" as $m \to \infty$ to the multi-valued function $\tilde{\theta}(t)$ defined by

$$(3.13)\qquad \tilde{\theta}(t) = \begin{cases} 1, & \text{in } t < 0 \\ [0,1], & \text{at } t = 0 \\ 0, & \text{in } t > 0. \end{cases}$$

Notice that they satisfy the following properties.

(1) $\theta''_m(t) = \theta'_m(t - 1/m)$,

(2) $\theta''_m(t) - \theta'_m(\tau) \leqq 0$, for $t - \tau > 1/m$,

(3) $\theta''_m(\tau) - \theta'_m(t) \geqq 0$, for $\tau < t$.

Relations (2) and (3) are immediate consequences of (1).

Let us denote by u'_m and u''_m, respectively, the solutions of the non-linear mixed boundary value problems defined by θ'_m and θ''_m. That is, denoting the quasi-linear form $b_\theta(u,v)$ corresponding to $\theta = \theta'_m$ and θ''_m respectively by $b'_m(u,v)$ and $b''_m(u,v)$, we have

$$u_m \in V;\ b'_m(u'_m, v) = \int_\Omega f v dx + \int_{\partial_2\Omega} g v d\sigma, \text{ for all } v \in V, \tag{3.14}$$

$$u''_m \in V;\ b''_m(u'_m, v) = \int_\Omega f v dx + \int_{\partial_2\Omega} g v d\sigma, \text{ for all } v \in V. \tag{3.15}$$

We know by Theorem (3.2) that u'_m, $u''_m \in V \cap C^{0,\lambda}(\bar{\Omega})$. Then we have the following two propositions.

PROPOSITION 3.3 *The sequence u'_m satisfying* (3.14) (*resp. u''_m satisfying* (3.15)) *is non-decreasing* (*resp. non-increasing*).

PROOF. If $m_1 < m_2$, then from (3.14) we obtain

$$a(u'_{m_2} - u'_{m_1}, v) = \int_\Omega P(\psi, f)[\theta'_{m_2}(u'_{m_2} - \psi) - \theta'_{m_1}(u'_{m_1} - \psi)]v dx$$

$$+ \int_{\partial_2\Omega} Q(\psi, g)[\theta'_{m_2}(u'_{m_2} - \psi) - \theta'_{m_1}(u'_{m_1} - \psi)]v d\sigma.$$

Here we can write

$$\begin{aligned}\theta'_{m_2}(u'_{m_2} - \psi) - \theta'_{m_1}(u'_{m_1} - \psi) &= \theta'_{m_2}(u'_{m_2} - \psi) - \theta'_{m_2}(u'_{m_1} - \psi) \\ &\quad + \theta'_{m_2}(u'_{m_1} - \psi) - \theta'_{m_1}(u'_{m_1} - \psi) \\ &\geqq \theta'_{m_2}(u'_{m_2} - \psi) - \theta'_{m_2}(u'_{m_1} - \psi)\end{aligned}$$

since $\theta'_{m_2}(t) \geqq \theta'_{m_1}(t)$ for all t because $m_2 > m_1$. The same holds also in the boundary integral on $\partial_2\Omega$ above. The function

$$v = \min(u'_{m_2} - u'_{m_1}, 0)$$

vanishes on the (relatively closed) subset E of $\bar{\Omega}$ where $u'_{m_2} \geqq u'_{m_1}$ and as $\partial_1\Omega \subset E$, it follows that v belongs to V. Since $v \leqq 0$ everywhere in $\bar{\Omega}$ and since, on the subset $\bar{\Omega} - E$, we have

$$\theta'_{m_2}(u'_{m_2} - \psi) - \theta'_{m_2}(u'_{m_1} - \psi) \geqq 0,$$

the above inequality implies that

$$[\theta'_{m_2}(u'_{m_2} - \psi) - \theta'_{m_1}(u'_{m_1} - \psi)]v \geqq 0.$$

Substituting this $v \in V$, we see that

$$\begin{aligned}a(v, v) &= a(u'_{m_2} - u'_{m_1}, v) \\ &= \int_{\Omega - E} P(\psi, f)[\theta'_{m_2}(u'_{m_2} - \psi) - \theta'_{m_1}(u'_{m_1} - \psi)]\, v dx \\ &\quad + \int_{\partial_2\Omega - E} Q(\psi, g)[\theta'_{m_2}(u'_{m_2} - \psi) - \theta'_{m_1}(u'_{m_1} - \psi)]\, v d\sigma \leqq 0.\end{aligned}$$

Then by the coercitivity of $a(u,v)$, it follows that v is a constant in $\bar{\Omega}$. But $v = 0$ on $\partial_1\Omega$. Hence by the connectedness of $\bar{\Omega}$, we see that $v = 0$ in Ω. In particular, $u'_{m_2} = u'_{m_1}$ on the subset where $u'_{m_2} < u'_{m_1}$ which is absurd. Hence $u'_{m_2} \geqq u'_{m_1}$ in $\bar{\Omega}$ for $m_2 > m_1$.

The proof of the assertion that $\{u''_m\}$ is non-increasing is completely analogous and if $u''_{m_2} > u''_{m_1}$ at some point of $\bar{\Omega}$, it is enough to take for v the function $v = \max(u''_{m_2} - u''_{m_1}, 0) \in V$ for $m_1 < m_2$.

This completes the proof.

The same method also yields the

PROPOSITION 3.4 *The sequences* $\{u'_m\}$ *and* $\{u''_m\}$ *respectively defined by* (3.14) *and* (3.15) *satisfy*

$$0 \leqq u''_m - u'_m \leqq 1/m, \text{ for each } m. \tag{3.16}$$

PROOF. We have, for any $v \in V$,

$$\begin{aligned} a(u''_m - u'_m, v) = & \int_\Omega P(\psi,f)\,[\theta''_m(u''_m - \psi) - \theta'_m(u'_m - \psi)]\,v\,dx \\ & + \int_{\partial_2\Omega} Q(\psi,g)\,[\theta''_m(u''_m - \psi) - \theta'_m(u'_m - \psi)]\,v\,d\sigma. \end{aligned}$$

Since u'_m and u''_m are continuous, if $u''_m < u'_m$ at some point of $\Omega \cup \partial_2\Omega$ then there exists a relatively open subset ω of $\bar{\Omega}$ where $u''_m(x) < u'_m(x)$. Let $v = \min(u''_m - u'_m, 0)$. Then as before $v \in V$, $v \leqq 0$ and $v = 0$ in $\Omega \cup \partial_2\Omega - \omega$ so that

$$\begin{aligned} a(v,v) = & \int_{\Omega\cap\omega} P(\psi,f)\,[\theta''_m(u''_m - \psi) - \theta'_m(u'_m - \psi)]\,(u''_m - u'_m)\,dx \\ & + \int_{\partial_2\Omega\cap\omega} Q(\psi,g)\,[\theta''_m(u''_m - \psi) - \theta'_m(u'_m - \psi)]\,(u''_m - u'_m)\,d\sigma. \end{aligned}$$

By the property (3) of the functions θ'_m and θ''_m we have

$$\theta''_m(u''_m - \psi) - \theta'_m(u'_m - \psi) \geqq 0 \text{ since } u''_m < u'_m \text{ on } \omega.$$

Therefore $a(v,v) \leqq 0$ and, by the coercivity, v is a constant in Ω. Since $v = 0$ on $\bar{\omega}$ it follows by the connectedness of $\bar{\Omega}$, as in the above proof, that $v = 0$ everywhere; that is, $u''_m = u'_m$ on ω which is absurd thus proving $u''_m \geqq u'_m$ for each m.

The proof of the assertion that $u''_m - u'_m \leqq 1/m$ is quite similar. It is enough to take $v = \max(u''_m - u'_m - 1/m, 0) \in V$ if $u''_m - u'_m > 1/m$ at some point and hence in some open subset of $\bar{\Omega}$ as before, and to use the property (1) of the functions θ'_m and θ''_m. This completes the proof of the proposition.

In view of Theorem (3.2), the two sequences $\{u'_m\}$, $\{u''_m\}$ form bounded sets in $V \cap C^{0,\lambda}(\bar{\Omega})$ and hence, in particular, have weakly convergent subsequences in $V \cap C^{0,\lambda}(\bar{\Omega})$. On the other hand, it follows by the above two pro positions that the sequences tend (weakly) to a common limit u in $V \cap C^{0,\lambda}(\bar{\Omega})$.

Thus $u \in V \cap C^{0,\lambda}(\bar{\Omega})$.

It remains to prove that u is the unique solution of the variational inequality (3.9).

For this purpose we consider the mixed boundary value problem (3.4) with $\theta = \theta'_m$ defined by (3.11) and we denote the corresponding unique solution (whose existence is assured by Theorem (3.1)) by u'_m. Then we have

THEOREM 3.5 *If the Assumptions A (or A'), B and C hold, then the solution u'_m of the mixed problem* (3.14) *satisfies $u'_m \geqq \psi - 1/m$.*

PROOF. Suppose, if possible, $u'_m(x) < \psi(x) - 1/m$ at some point $x \in \Omega \cup \partial_2\Omega$. Then under the assumptions made, u'_m belongs to $C^{0,\lambda}(\bar{\Omega}) \cap V$ by Theorem (3.2). Hence there exists a non-empty open subset ω of $\Omega \cup \partial_2\Omega$ where $u'_m < \psi - 1/m$. Then the function

$$v = \min(u'_m - \psi + 1/m, 0) = \begin{cases} u'_m - \psi + 1/m & \text{in } \omega \\ 0 & \text{in } \Omega \cup \partial_2\Omega - \omega \end{cases}$$

belongs to V since on $\partial_1\Omega$ we have $u'_m = 0$ and $\psi \leqq 0$. Substituting v in the equation

$$b'_m(u'_m, v) = \int_\Omega f v dx + \int_{\partial_2\Omega} g v d\sigma,$$

we obtain

$$a(u'_m, v) = \int_{\omega\cap\Omega} [P(\psi,f)\theta'_m(u'_m - \psi) + f] v dx + \int_{\omega\cap\partial_2\Omega} [Q(\psi,g)\theta'_m(u'_m - \psi) + g] v d\sigma.$$

On the other hand, we also have

$$a(\psi, v) = \int_{\omega\cap\Omega} (A\psi) v dx + \int_{\omega\cap\partial_2\Omega} (\partial\psi/\partial\nu) v d\sigma$$

and so, on subtraction,

$$a(u'_m - \psi, v) = \int_{\omega\cap\Omega} [P(\psi,f)\theta'_m(u'_m - \psi) + f - A\psi] v dx + \int_{\omega\cap\partial_2\Omega} [Q(\psi,g)\theta'_m(u'_m - \psi) + g - (\partial\psi/\partial\nu)] v d\sigma.$$

Since $u'_m - \psi < -1/m$ in ω and $\theta'_m(t) = 1$ in $t \leqq -1/m$, it follows that $a(u'_m - \psi, v) \leqq 0$, that is, $a(v,v) \leqq 0$. Hence by the V-coercitivity of the bilinear form $a(u,v)$, we find that $v =$ constant on ω. But, since $v = 0$ at least at one point of $\bar{\omega}$, it follows that $v = 0$ in ω by the connectedness of Ω. This means that $u'_m = \psi - 1/m$ in ω which contradicts our assumption. Hence $u'_m \geqq \psi - 1/m$ everywhere in Ω, which proves the required assertion.

On the other hand, we have already seen that u'_m form a bounded set in $V \cap C^{0,\lambda}(\bar{\Omega})$ (by Theorem (3.2)). Hence, by weak compactness, a subsequence, again denoted by u'_m, converges weakly in $V \cap C^{0,\lambda}(\bar{\Omega})$ and, moreover, uniformly to u. This means that $u \geqq \psi$ in Ω, that is, $u \in \mathbf{K}$.

We are now in a position to prove the first main result of this section.

THEOREM 3.6 *If the Assumptions A (or A'), B and C are satisfied, then u is a solution of the variational inequality* (3.9).

PROOF. We have already shown above that $u \in \mathbf{K}$. If $v \in \mathbf{K}$, then $v - u'_m \in V$ for each m (in the subsequence). Since the quasi-linear form $b'_m(u,v)$ (corresponding to the function θ'_m) is monotone and (hemi-) continuous it follows, on applying Minty's lemma (see for instance [22]), that

$$\begin{aligned} b'_m(v, v - u'_m) &\geqq b'_m(u'_m, v - u'_m) \\ &= a(u'_m, v - u'_m) - \int_\Omega P(\psi,f)\theta'_m(u'_m - \psi)(v - u'_m)\,dx \\ &\quad - \int_{\partial_2\Omega} Q(\psi,g)\theta'_m(u'_m - \psi)(v - u'_m)d\sigma \\ &= \int_\Omega f(v - u'_m)dx + \int_{\partial_2\Omega} g(v - u'_m)d\sigma. \end{aligned}$$

Since $v \in \mathbf{K}$ implies that $v - \psi \geqq 0$ so that $\theta'_m(v - \psi) = 0$, we have

$$P(\psi,f)\theta'_m(v - \psi) = 0 \text{ in } \Omega, \; Q(\psi,g)\theta'_m(v - \psi) = 0 \text{ on } \partial_2\Omega,$$

and hence

$$a(v, v - u'_m) = b'_m(v, v - u'_m) \text{ for } v \in \mathbf{K}.$$

We thus obtain the inequality

$$a(v, v - u'_m) \geqq \int_\Omega f(v - u'_m)dx + \int_{\partial_2\Omega} g(v - u'_m)\,d\sigma.$$

Here since $u'_m \to u$ weakly in V, we can pass to the limits on both sides and we find that

$$a(v, v - u) \geqq \int_\Omega f(v - u)dx + \int_{\partial_2\Omega} g(v - u)\,d\sigma.$$

In order to complete the proof of the theorem, it is sufficient to apply the lemma of Minty to the bilinear form $a(u,v)$ to conclude from the above inequality that

$$a(u, v-u) \geqq \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma, \text{ for all } v \in \mathbf{K}.$$

Theorems (3.5) and (3.6) provide a method of obtaining certain regularity results for the solution of the variational inequality (3.9) from the corresponding results for solutions of linear mixed boundary value problems. We examine some of these cases in the following.

Consider for any $F \in L^p(\Omega)$ with $p > n/2$ and $G \in L^q(\partial_2\Omega)$ with $q > n-1$ the mixed boundary value problem

$$\text{(3.17)} \qquad w \in V; \ a(w,v) = \int_\Omega Fv dx + \int_{\partial_2\Omega} Gv d\sigma, \text{ for all } v \in V.$$

It is known that the regularity of the solution w near a point x in $\bar{\Omega}$ varies according as whether the point x under consideration belongs to one of the following sets:

$$(\alpha)\ \Omega \cup \partial_1\Omega, \quad (\beta)\ \partial_2\Omega, \quad (\gamma)\ \overline{\partial_1\Omega} \cap \overline{\partial_2\Omega}.$$

We make the following hypotheses which require suitable smoothness conditions on the domain Ω (respectively, subdomains of Ω) and the coefficients of the operator A. In fact, we shall implicitly assume that the boundary $\partial\Omega$ and the coefficients a_{jk} of the operator are sufficiently smooth in order that the a priori estimates that we recall below are valid for w.

In order to state these a priori estimates, we shall require the fractionary Sobolev spaces $W^{s,p}(\Omega)$ and $W^{s,p}(\partial_2\Omega)$ for any real s. For details concerning the definitions and properties of these spaces, we refer to the paper of Lions and Magenes [9] or to the paper of Shamir [17]. We shall denote the norm in the space $W^{s,p}(\Omega)$ by $\|\cdot\|_{s,p,\Omega}$.

Further, in view of the imbedding theorem for fractionary Sobolev spaces, we know that (when $\partial_2\Omega$ is smooth) for $0 < s < 1$, $W^{s,l}(\partial_2\Omega) \subset L^r(\partial_2\Omega)$ where $(1/r) = (1/l) - s/(n-1)$ and the inclusion mapping is continuous. Then by duality, the functions in $L^{r'}(\partial_2\Omega)$ $((1/r) + (1/r') = 1)$ define continuous linear functionals on $W^{s,l}(\partial_2\Omega)$, that is, define elements of $W^{-s,l'}(\partial_2\Omega)$, $(1/l) + (1/l') = 1$.

We shall consider the subdomains Ω', Ω'' of Ω which are of the form $\Omega' = \Omega \cap B'$, $\Omega'' = \Omega \cap B''$ where B' and B'' are open balls in $\mathbf{R}^n$ with centre at x such that $\bar{B}' \subset B''$.

Starting with case (α) that $x \in \Omega \cup \partial_1\Omega$ by taking Ω'' such that $\bar{\Omega}'' \subset \Omega \cup \partial_1\Omega$, we have the estimate

$$(3.18) \qquad \| w \|_{2,p,\Omega'} + \| w \|_V \leqq \text{const.} \{\| F \|_{p,\Omega''} + \| w \|_{p,\Omega''}\}.$$

The a priori estimate (3.18) is a consequence of the results due to Agmon, Douglis and Nirenberg [1] on elliptic boundary value problems.

In case (β), that is, for $x \in \partial_2\Omega$, by taking Ω'' such that $\bar{\Omega}'' \subset \Omega \cup \partial_2\Omega$ we have the estimate

$$(3.19) \quad \| w \|_{1+(1/p),p,\Omega'} + \| w \|_V \leqq \text{const.} \{\| F \|_{p,\Omega''} + \| G \|_{q,\bar{\Omega}''\cap\partial_2\Omega} + \| w \|_{p,\Omega''}\},$$

where $q \leqq \max(p, n-1)$.

The estimate (3.19) is a consequence of the results of Lions and Magenes [9] on inhomogeneous boundary value problems.

In the remaining case (γ) where $x \in \overline{\partial_1\Omega} \cap \overline{\partial_2\Omega}$, we have the following estimate from the results of Shamir [17].

$$(3.20) \qquad \| w \|_{s,p,\Omega''} + \| w \|_V \leqq \text{const.} \{\| F \|_{p,\Omega''} + \| G \|_{q,\bar{\Omega}''\cap\partial_2\Omega} + \| w \|_{p,\Omega''}\}$$

where

(i) $s = 1$ for $1 < p < 4$,

(ii) $s < (\frac{1}{2}) + (2/p)$ for any $p > 4$

and $q = p(n-1)/n$ so that $p > n$ implies that $q > n-1$.

We remark that in [17] an estimate of the type (3.20) is proved with $\| G \|_{-1/p,p,\bar{\Omega}''\cap\partial_2\Omega}$ instead of $\| G \|_{q,\bar{\Omega}''\cap\partial_2\Omega}$ as stated here. However, making use of the remark made earlier about the fractionary Sobolev spaces, we can deduce (3.20) from the a priori estimate of Shamir in [17].

Assume that the domain Ω satisfies Assumptions A (or A′) and B. Let $f \in L^p(\Omega)$ with $p > n/2$ and $g \in L^q(\partial_2\Omega)$ with $q > n-1$ be given. Suppose that the obstacle ψ and the coefficients of A satisfy the Assumption C.

In order to derive the regularity of the solution of the variational inequality from the above estimates, we consider the sequence of Lipschitz functions θ'_m defined by (3.11) and denote by u_m the solution of the corresponding non-linear mixed problem (3.14)

$$b'_m(u_m, v) = \int_\Omega f v dx + \int_{\partial_2\Omega} g v d\sigma, \text{ for all } v \in V.$$

Then $u_m \in W^{s,p}(\Omega') \cap V$ for appropriate s, p, q according as the estimate (3.18), (3.19) or (3.20) holds and satisfies the following estimate.

$$\| u_m \|_V + \| u_m \|_{s,p,\Omega'} \leqq C(\Omega', \Omega'', p, q) \{ \| P(\psi, f) \|_{p,\Omega''} + \| f \|_{p\,\Omega''}$$

$$+ \| u_m \|_{p,\Omega''} + \| Q(\psi, g) \|_{q\,\partial_2\Omega \cap \bar{\Omega}''} + \| g \|_{q,\partial_2\Omega \cap \bar{\Omega}''} \}.$$

On the other hand we know that $\| u_m \|_{p,\Omega''}$ can be estimated by the other terms on the right side (see Murthy and Stampacchia [15]). From (3.18), (3.19) and (3.20), we conclude that u_m forms a bounded sequence in $W^{s,p}(\Omega') \cap V$ for the appropriate s, p. So, by the weak relative compactness (of bounded sets), a subsequence, again denoted by u_m, converges weakly to a limit u in $W^{s,p}(\Omega') \cap V$. But, by Theorems (3.5) and (3.6), this limit u is the solution of the variational inequality (3.9). Thus we have proved the following main result on the regularity of the solution of the variational inequality.

THEOREM 3.7 *Let Ω satisfy the Assumptions A (or A′) and B. Suppose given $f \in L^p(\Omega)$ with $p > n/2$ and $g \in L^q(\partial_2\Omega)$ with $q > n - 1$ and suppose that the obstacle ψ and the coefficients of A satisfy Assumption C. If $u \in \mathbf{K}$ is the solution of the variational inequality* (3.9) *and if $x \in \Omega \cup \partial_1\Omega$ (resp. $x \in \partial_2\Omega$, $x \in \overline{\partial_1\Omega} \cap \overline{\partial_2\Omega}$), then $u \in H^{2,p}(\Omega') \cap V$ (resp. $W^{1+(1/p),p}(\Omega') \cap V$, $W^{s,p}(\Omega') \cap V$).*

As a consequence of the fact that we have assumed that the coefficients of the operator A and the domain Ω (resp. subdomains of Ω) are suitably smooth, Theorem 3.7 yields, in view of the Sobolev inequalities, the following

COROLLARY 3.8 *The solution $u \in \mathbf{K}$ of the variational inequality* (3.9) *belongs to $C^{1,\mu}(K)$ for any compact subset K of $\bar{\Omega}$ such that $K \subset \Omega \cup \partial_1\Omega$ with $\mu = 1-(n/p)$. Moreover, Theorem 2.9 can be strenghthened for $p=q=\infty$, in the sense that u is in $C^{0,\lambda}(K)$ with $0 < \lambda < \frac{1}{2}$ for any compact subset K of $\bar{\Omega}$ while λ is any number less than 1 provided, in addition, that $K \cap \overline{\partial_1\Omega} \cap \overline{\partial_2\Omega} = \emptyset$.*

4. Interpretation of the boundary data and some remarks

In the first part of this section, we give an interpretation of the boundary conditions formally imposed by the variational inequality (3.9). In the rest of the section, we make a few remarks concerning extensions and generalizations of our results of the previous sections.

I. We recall that the sequence of Lipschitz functions θ'_m defined by (3.11) "converge" to the multi-valued function $\tilde{\theta}$ defined by (3.13). On the other hand, under Assumptions A (or A′), B and C, Theorems (3.5) and (3.6) show that the solutions u'_m (a subsequence of u'_m) of the non-linear mixed boundary value problems converge in $V \cap C^{0,\lambda}(\bar{\Omega})$ to the solution of the variational inequality (3.9). Thus the variational inequality (3.9) can be formally described as follows:

$$
(4.1) \quad \begin{cases} Au - f \in \max(A\psi - f, 0)\,\tilde{\theta}\,(u - \psi) \text{ in } \Omega\,, \\ u = 0 \text{ on } \partial_1\Omega,\ \partial u/\partial v - g \in \max((\partial\psi/\partial v) - g, 0)\tilde{\theta}(u - \psi) \text{ on } \partial_2\Omega. \end{cases}
$$

We observe that if ω is an open subset of $\bar{\Omega}$ where $u > \psi$, then $\tilde{\theta}(u - \psi) = 0$ and so u is a solution of the linear mixed boundary value problem

$$
(4.2) \quad \begin{cases} Au = f \text{ in } \omega \cap \Omega \text{ (in the sense of distributions)}\,, \\ u = 0 \text{ on } \omega \cap \partial_1\Omega,\ \partial u/\partial v = g \text{ on } \omega \cap \partial_2\Omega. \end{cases}
$$

In order to interpret problem (3.1), we find, on taking $v \in C_0^\infty(\Omega)$, that $Au = f$ in Ω (in the sense of distributions). Let $D(A)$ denote the subspace of V consisting of all $u \in V$ such that Au, taken in the sense of distributions, belong to $L^2(\Omega)$. (We note that if the coefficients of A are functions in $C^1(\bar{\Omega})$, then $C^2(\bar{\Omega}) \cap V$ is dense in $D(A)$).

If $\partial_2\Omega$ is of class C^1, then it admits a continuously varying tangent space at each of its points and a continuous normal vector field v_0 oriented towards the interior of Ω. Then, for any $u \in C^1(\bar{\Omega}) \cap D(A)$, we obtain by applying Green's formula

$$
(4.3) \quad \int_\Omega (Au)v\,dx = a(u,v) - \int_{\partial_2\Omega} \frac{\partial u}{\partial v} v\,d\sigma
$$

where

$$
(4.4) \quad \frac{\partial u}{\partial v} = a_{jk}(x)v_k(x)u_{x_j}.
$$

Here v_k $(k = 1, \cdots, n)$ denote the direction cosines of the interior normal $v_0(x)$ at x on $\partial_2\Omega$. $\partial u/\partial v$ is called the co-normal derivative of u with respect to the operator A. Thus we see that if $u \in C^1(\bar{\Omega}) \cap D(A)$, then

$$
(4.5) \quad a(u,v) = \int_\Omega f v\,dx + \int_{\partial_2\Omega} (\partial u/\partial v)v\,d\sigma, \text{ for all } v \in V.
$$

Now suppose that $u \in V$ is arbitrary and $\partial_2\Omega$ is locally Lipschitz. Then $\partial u/\partial v$ can still be defined in a generalized sense as follows.

Let $V(\partial_2\Omega) = V/V_0$ (V_0 being the space of all functions v in V having its trace on $\partial_2\Omega$ zero) be provided with the quotient norm. Then the mapping which associates to every $v \in V$ its trace $v \mid \partial_2\Omega$ is continuous linear from V onto $V(\partial_2\Omega)$. On the other hand, the mapping L defined by

$$Lv = \langle Au, v\rangle - a(u, v)$$

defines a continuous linear functional on V which is zero on V_0 and hence defines a continuous linear functional on the quotient space $V(\partial_2\Omega)$. In other words, there exists a unique element $G(u) \in [V(\partial_2\Omega)]'$, the dual space of $V(\partial_2\Omega)$, such that

$$\langle G(u), v\rangle = \langle Au, v\rangle - a(u, v). \tag{4.6}$$

This can be considered as an extension of Green's formula (4.5) above. By definition we set

$$(\partial u / \partial v) = G(u) \text{ on } \partial_2\Omega. \tag{4.7}$$

According to what we said at the beginning, we know that $V(\partial_2\Omega) \subset L^s(\partial_2\Omega)$ and the inclusion mapping is continuous so that every $g \in L^{s'}(\partial_2\Omega)$ defines a continuous linear functional on $V(\partial_2\Omega)$. Moreover, we can then write

$$\langle G(u), v\rangle = \int_{\partial_2\Omega} gv d\sigma.$$

That is,

$$(\partial u / \partial v) = g \text{ on } \partial_2\Omega \text{ in a ``generalized sense''}. \tag{4.8}$$

A detailed account of these facts can be found, for instance, in the book of Lions and Magenes [10] or in the article of Magenes and Stampacchia [12].

These considerations lead us to the following formal interpretation of the boundary conditions.

1) If there exists an open subset E_1 of $\partial_2\Omega$ where $u > \psi$, then $\partial u / \partial v = g$ on E_1.

2) If $u = \psi$ and $g - \partial\psi / \partial v$ is a positive measure on a subset E_2 of $\partial_2\Omega$, then again $\partial u / \partial v = g$ on E_2.

3) If $u = \psi$ and $\partial\psi / \partial v - g$ is a positive measure on a subset E_3 of $\partial_2\Omega$ then, since $0 \leqq \tilde{\theta}(t) \leqq 1$, we have

$$g \leqq \partial u / \partial v \leqq \partial\psi / \partial v \text{ on } E_3.$$

These inequalities are to be understood in a generalized sense. In order to make these more precise, we note that the notion "$v \geqq 0$ on a subset E of $\bar{\Omega}$" for functions $v \in V$ induces a notion of positivity on $V(\partial_2\Omega)$ which is the quotient space V/V_0. We recall that $V(\partial_2\Omega)$ is precisely the space of traces on $\partial_2\Omega$ of elements of V.

Then for elements G in the dual space $[V(\partial_2\Omega)]'$, we define positivity in a natural way as follows: For $G \in [V(\partial_2\Omega)]'$ and for a subset E of $\partial_2\Omega$ we say that "$G \geqq 0$ on E" if $G(v) \geqq 0$ for all $v \in V(\partial_2\Omega)$ such that "$v \geqq 0$ on E".

An analogous definition can also be given to the elements of $[H^1(\partial_2\Omega)]' = (H^1(\Omega)/V_0)'$. We also observe that as $V(\partial_2\Omega) \subset H^1(\partial_2\Omega)$ with continuous inclusion, the elements of $[H^1(\partial_2\Omega)]'$ define elements of $[V(\partial_2\Omega)]'$ (by composition with this inclusion) and we identify these functionals.

In view of this, the inequalities in (3) above are to be taken in the sense of functionals in $[V(\partial_2\Omega)]'$, namely,

$$(\partial u/\partial v) - g \geqq 0 \text{ and } (\partial\psi/\partial v) - (\partial u/\partial v) \geqq 0 \text{ on } E_3$$

as elements of $[V(\partial_2\Omega)]'$.

Since $[V(\partial_2\Omega)]'$ is a normal space of distributions on $\partial_2\Omega$ we see, by a well known theorem of Riesz-Schwartz, that the above inequalities can also be understood in the sense of measures on $\partial_2\Omega$; that is, $(\partial u/\partial v) - g$ and $(\partial\psi/\partial v) - (\partial u/\partial v)$ are positive measures on E_3.

II. The solution u of the variational inequality (3.9) can also be obtained by another approximation procedure of potential theoretic nature. Again here we essentially follow the treatment of Lewy and Stampacchia in [7].

Suppose $u \in \mathbf{K}$ is the solution of the variational inequality (3.9). Let $\mathbf{K}_u$ denote the cone of all $w \in V$ which can be written in the form $w = t(u - v)$ for some $v \in \mathbf{K}$ and $t > 0$, and $\bar{\mathbf{K}}_u$ be its closure in V. Then it is clear that

$$a(u, w) \geqq \int_\Omega fw\,dx + \int_{\partial_2\Omega} gw\,d\sigma, \text{ for all } w \in \bar{\mathbf{K}}_u. \tag{4.9}$$

We next observe that the positive cone $\{w \in V;\ w \geqq 0 \text{ in } \bar{\Omega}\}$ is contained in $\bar{\mathbf{K}}_u$ and in particular, (4.9) is satisfied. These considerations lead us to introduce, in analogy with the case of the Dirichlet problem (that is, $\partial_1\Omega = \partial\Omega$), the following

Definition. A distribution $w \in H^1(\Omega)$ is said to be a super solution with respect to V, A, f and g if

$$a(w, \phi) \geqq \int_\Omega f\phi\,dx + \int_{\partial_2\Omega} g\phi\,d\sigma, \text{ for all } \phi \in C^1(\bar{\Omega}) \text{ with } \phi = 0 \text{ on } \partial_1\Omega \text{ and } \phi \geqq 0 \text{ in } \bar{\Omega}. \tag{4.10}$$

Evidently in this definition, we can also take $\phi \in V$ with $\phi \geqq 0$ in $\bar{\Omega}$.

We have

THEOREM 4.1 *If* $u \in \mathbf{K}$ *is the solution of the variational inequality* (3.9) *and* W *denotes the set of all super-solutions with respect to* V, A, f *and* g *such that*

$$w \geqq 0 \text{ on } \partial_1\Omega \text{ and } w \geqq \psi \text{ in } \Omega \tag{4.11}$$

then

$$u = \min\{w;\, w \in W\}. \tag{4.12}$$

PROOF. Let $w \in W$ be arbitrary and let $v = \min(u, w)$. Then $v \in \mathbf{K}$ because of (4.11) and we shall show that $v = u$. Substituting v in the variational inequality we get

$$a(u, v-u) \geqq \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma.$$

Since w is a super solution and $v - u \in V$ with $v - u \leqq 0$ in $\bar{\Omega}$, we have

$$a(w, v-u) \leqq \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma.$$

We can write the left hand side as

$$a(w, v-u) = \left(\int_{(u=w)} + \int_{(u>w)}\right) a_{jk} w_{x_k}(v-u)_{x_j} dx$$

where we have

$$v - u = 0 \text{ and } v_x = u_x \text{ on the set } \{x \in \bar{\Omega};\, u(x) = w(x)\}$$

and $v = w$ and $v_x = w_x$ on the set $\{x \in \bar{\Omega};\, u(x) > w(x)\}$. Hence the first integral vanishes and

$$a(v, v-u) = a(w, v-u) \leqq \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma. \tag{4.13}$$

The inequalities (3.9) and (4.13) together imply that $a(v-u, v-u) \leqq 0$ which by V-coercivity of the bilinear form implies that $v - u$ is a constant in $\bar{\Omega}$. As in the proof of Theorem (3.5), the connectedness of $\bar{\Omega}$ shows that $v = u$ which proves the required assertion.

REMARK. Since any $\phi \in V$ such that $\phi \geqq 0$ in $\bar{\Omega}$ can always be written as $v - u$ with $v \in \mathbf{K}$, we see that the solution $u \in \mathbf{K}$ of the variational inequality (3.9) is itself a super solution with respect to V, A, f and g.

As a consequence of Theorem (4.1), we obtain

COROLLARY 4.2 *Let* ψ_1, ψ_2 *be two obstacles such that* $\psi_1 \geqq \psi_2$ *and* $\psi_j \leqq 0$ *on* $\partial_1\Omega$. *If* u_1, u_2 *are the two solutions of the variational inequality* (4.10) *corresponding to the convex sets*

$$\mathbf{K}_j = \{v \in V;\ v \geqq \psi_j\},\ (j = 1,2), \tag{4.14}$$

then $u_1 \geqq u_2$.

In fact, let W_j denote the two sets of super solutions with respect to V, F and A corresponding to $\psi_j (j = 1,2)$. Then $W_1 \subset W_2$ and by Theorem 4.1 we have

$$u_2 = \min_{w \in W_2} w \leqq \min_{w \in W_1} w = u_1,$$

which proves the assertion.

In conclusion we shall only mention that exactly as in Lewy and Stampacchia [8], one can prove the following assertions.

a) Let ψ_j be two (smooth) obstacles as in Corollary 4.2 and $u_j \in \mathbf{K}_j$ be the corresponding solutions of the variational inequality (3.9) $(j = 1,2)$. If there exists a point $x_0 \in \bar{\Omega}$ where $\psi_1(x_0) > 0$, then $\sup(u_1 - u_2)$ lies on the set

$$\{x \in \bar{\Omega};\ u_1(x) = \psi_1(x)\}.$$

b) If ψ_j and $u_j \in \mathbf{K}_j$ are as above, then

$$0 \leqq u_1 - u_2 \leqq \sup_{\bar{\Omega}} (\psi_1 - \psi_2).$$

c) If ψ_1 and ψ_2 are two smooth obstacles such that $\psi_j \leqq 0$ on $\partial_1\Omega$ and $u_j \in \mathbf{K}_j$, the corresponding solutions of the variational inequality, then

$$|u_1 - u_2| \leqq \sup_{\bar{\Omega}} |\psi_1 - \psi_2|.$$

The following are some extensions of our results.

(a) *Inhomogeneous data on* $\partial_1\Omega$. Let u_0 and ψ be two functions belonging to $H^1(\Omega)$ such that $\psi \leqq 0$ on $\partial_1\Omega$. Consider the closed convex set $\mathbf{K}_0$ in $H^1(\Omega)$ defined by

$$\mathbf{K}_0 = \{v \in H^1(\Omega);\ v - u_0 \in V \text{ and } v - u_0 \geqq \psi \text{ in } \Omega\}. \tag{4.15}$$

($\mathbf{K}_0$ is contained in the hyper-plane $V + u_0$ in $H^1(\Omega)$). Then all our results can be extended to the variational inequality

$$u \in \mathbf{K}_0;\ a(u, v - u) \geqq \int_\Omega [f_0(v - u) + f_j(v - u)_{x_j}]dx + \int_{\partial_2\Omega} g(v - u)d\sigma, \quad \text{for all } v \in \mathbf{K}_0, \tag{4.16}$$

with almost no change in the proofs by considering the convex set $\mathbf{K} = \mathbf{K}_0 + u_0$. The variational inequality (4.16) formally corresponds to the mixed boundary value problem

$$\text{(4.17)} \qquad \begin{cases} Aw = f_0 - (f_j)_{x_j} \text{ in } \Omega \text{ (in the sense of distributions)} \\ w = u_0 \text{ on } \partial_1\Omega,\ \partial w/\partial \nu = g \text{ on } \partial_2\Omega. \end{cases}$$

(b) *Operators with lower order terms.* Consider a uniformly elliptic operator of the form

$$\text{(4.18)} \qquad Au = -(a_{jk}u_{x_k} + d_ju)_{x_j} + b_ju_{x_j} + cu$$

where

$$\text{(4.19)} \qquad \begin{cases} \text{(i)} \quad a_{jk} \in L^\infty(\Omega),\ b_j, d_j \in L^{n+\varepsilon}(\Omega) \text{ and } c \in L^{n/2+\varepsilon}(\Omega) \text{ for some } \varepsilon > 0; \\ \text{(ii)} \quad \text{there exist positive constants } m, M \text{ such that} \\ \qquad m|\xi|^2 \leqq a_{jk}\xi_j\xi_k \leqq M|\xi|^2, \text{ a.e. in } \bar{\Omega} \text{ and for all } \xi \in \mathbf{R}^n - \{0\}; \\ \text{(iii)} \quad c - (d_j)_{x_j} \geqq c_0 > 0 \text{ on } \Omega \text{ in the sense of distributions } (c_0 \text{ a constant}). \end{cases}$$

We define the associated bilinear form by

$$\text{(4.20)} \qquad a(u,v) = \int_\Omega [(a_{jk}u_{x_k} + d_ju)v_{x_j} + (b_ju_{x_j} + cu)v]dx$$

and assume that $a(u,v)$ is V-coercive. Let $\psi \in H^1(\Omega)$ with $\psi \leqq 0$ on $\partial_1\Omega$ and $\mathbf{K} = \{v \in V;\ v \geqq \psi \text{ in } \Omega)$ be the associated closed convex set of V.

All the results of the previous sections easily extend with minor changes to the solutions of the variational inequality

$$u \in \mathbf{K};\ a(u, v-u) \geqq \int_\Omega [f_0(v-u) + f_j(v-u)_{x_j}]dx + \int_{\partial_2\Omega} g(v-u)\,d\sigma, \quad \text{for all } v \in \mathbf{K}.$$

If $a(u,v)$ is only semi-coercive on V (in the sense of Lions and Stampacchia [11]), it is necessary to make the compatibility assumption that

$$\int_\Omega f_0 dx + \int_{\partial_2\Omega} g d\sigma \leqq 0.$$

For the techniques we have used in the proof of Hölder continuity as applied to this more general situation, we refer to the paper of Stampacchia [19] (see also

Murthy and Stampacchia [15]). Clearly the remark (a) on the inhomogeneous data on $\partial_1\Omega$ applies also in this case.

(c) *Obstacles defined only on the boundary.* Let the domain Ω satisfy the conditions A (or A') and B of Section 1. We shall further assume that Ω and the coefficients a_{jk} of A are smooth in order that the inhomogeneous Dirichlet problem

$$(4.21) \qquad Aw = f \text{ in } \Omega, \ w = w_0 \text{ on } \partial\Omega,$$

that is,

$$(4.21)' \qquad w - w_0 \in H_0^1(\Omega), \ a(w,\eta) = \int_\Omega f\eta dx, \text{ for all } \eta \in H_0^1(\Omega)$$

is solvable and $w \in H^{2,p}(\Omega)$ for any given $f \in L^p(\Omega)$ and $w_0 \in H^{2,p}(\Omega)$.

This condition is satisfied, for instance, if $\partial\Omega$ is of class C^2 and $a_{jk} \in C^1(\Omega)$ (see the paper of Agmon, Douglis and Nirenberg [1]). Under the assumption of smoothness of $\partial\Omega$, the Dirichlet condition can equivalently be assigned in the appropriate space of distributions on $\partial\Omega$ itself (see the paper of Lions and Magenes [9]).

Let ψ be a distribution belonging to $W^{2-(1/p),p}(\partial\Omega)$ such that $\psi \leqq 0$ on $\partial_1\Omega$, $f \in L^p(\Omega)$ with $p > n$ and let $g \in L^q(\partial_2\Omega)$ with $q > n-1$. Let $\tilde{\psi} \in H^{2,p}(\Omega)$ be the (unique) solution of the Dirichlet problem.

$$(4.22) \qquad A\tilde{\psi} = f \text{ in } \Omega, \ \tilde{\psi} = \psi \text{ on } \partial\Omega.$$

Consider the closed convex subsets of V defined by ψ and $\tilde{\psi}$, namely,

$$(4.23) \qquad \mathbf{K} = \{v \in V;\, v \geqq \psi \text{ on } \partial\Omega\} \text{ and } \tilde{\mathbf{K}} = \{v \in V;\, v \geqq \tilde{\psi} \text{ in } \Omega\}.$$

We observe that $\tilde{\psi}$ satisfies Assumption C of Section 2 and, by Sobolev's inequality, $\tilde{\psi}$ belongs to $C^{1,\gamma}(\bar{\Omega})$, $\gamma = 1 - n/p$. Then we have the

THEOREM 4.3 *Under the above assumptions if u is the solution of the variational inequality*

$$(4.24) \quad u \in \tilde{\mathbf{K}};\, a(u, v-u) \geqq \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma, \text{ for all } v \in \tilde{\mathbf{K}},$$

then u resolves the variational inequality

$$(4.25) \quad u \in \mathbf{K};\, a(u, v-u) \geqq \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma, \text{ for all } v \in \mathbf{K}.$$

PROOF. We note, first of all, that $u \in \mathbf{K}$ in view of the inclusion $\tilde{\mathbf{K}} \subset \mathbf{K}$. It is therefore enough to show that (4.24) holds for all $v \in \mathbf{K}$ so that (4.25) is satisfied.

If $v \in \mathbf{K}$ is any arbitrary element, then either $v \geqq \tilde{\psi}$ in Ω or $v < \tilde{\psi}$ at some point in Ω (and hence also in an open subset of Ω by the continuity of $v - \tilde{\psi}$ given by Theorem (2.9)). In the first case, (4.24) is nothing but (4.25). So we have only to consider the case in which $v < \tilde{\psi}$ at some point of Ω.

In this case, we can write v as a sum $v = v_1 + v_2$ by defining

$$v_1 = \max(v, \tilde{\psi}) \text{ and } v_2 = v - v_1. \tag{4.26}$$

Then $v_1 \in \tilde{\mathbf{K}}$ and $v_2 \in V$ with

$$v_2 = \begin{cases} 0 & \text{in } \{x \in \bar{\Omega};\, v \geqq \tilde{\psi}\} \\ v - \tilde{\psi} & \text{in } \{x \in \bar{\Omega};\, v < \tilde{\psi}\}. \end{cases}$$

Since $v \in \mathbf{K}$ by assumption so that $v \geqq \psi = \tilde{\psi}$ on $\partial\Omega$, it follows that supp $v_2 \subset \Omega$ and thus $v_2 \in H_0^1(\Omega) \subset V$. We can now write

$$a(u, v - u) = a(u, v_1 - u) + a(u, v_2). \tag{4.27}$$

We shall show that

$$a(u, v_2) = \int_\Omega f v_2 dx. \tag{4.28}$$

In fact, by Theorems 3.5 and 3.6, u is the weak limit in $V \cap C^{0,\lambda}(\bar{\Omega})$ of the sequence (a subsequence) u'_m which are solutions of the mixed boundary value problems

$$u'_m \in V;\; a(u'_m, \eta) = \int_\Omega [\max(A\tilde{\psi} - f, 0)\theta'_m(u'_m - \psi) + f]\eta dx$$
$$+ \int_{\partial_2\Omega} (\max((\partial\psi/\partial\nu) - g, 0)\theta'_m(u'_m - \psi) + g]\eta d\sigma, \text{for all } \eta \in V.$$

Here, since $A\tilde{\psi} = f$ in Ω and since supp $v_2 \subset \Omega$ we find that

$$a(u'_m, v_2) = \int_\Omega f v_2 dx,$$

which on passage to the limit proves (4.28).

Finally, since u is the solution of (4.24) and since $v_1 \in \tilde{\mathbf{K}}$, we find from (4.27) that

$$a(u, v - u) \geqq \int_\Omega f(v_1 - u)dx + \int_{\partial_2\Omega} g(v_1 - u)d\sigma + \int_\Omega f v_2 dx.$$

In view of the fact that supp $v_2 \subset \Omega$ implies that $v = v_1$ on $\partial\Omega$, this inequality proves the required assertion. The method of proof we have adopted above follows an idea of Kinderlehrer [5]. The remarks (a) and (b) can easily be extended to this situation.

Theorem (4.3), in particular, together with our results of Section 2 give the results of Da Veiga [23] and Brézis [2]. However, the proof of Da Veiga being direct does not require the smoothness of $\partial\Omega$ nor that of the coefficients as indicated here.

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SCUOLA NORMALE SUPERIORE, PISA
AND
TATA INSTITUTE OF FUNDAMENTAL RESEARCH, BOMBAY